IQI 04, Seminar 5

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- Continuous one-qubit rotations.
- Application: Refocusing.
- Conditional rotations.
- Phase kick-back.
- The rotation-angle problem.

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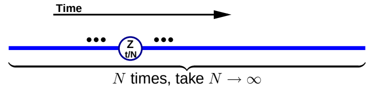
Continuous Z-Rotation

- $\mathbf{Z}_t$ defines a one-parameter group:

$$\begin{array}{ccc} \mathbf{Z}_s & \mathbf{Z}_t & = & \mathbf{Z}_{s+t} \\ \begin{pmatrix} e^{-is/2} & 0 \\ 0 & e^{is/2} \end{pmatrix} & \begin{pmatrix} e^{-it/2} & 0 \\ 0 & e^{it/2} \end{pmatrix} & = & \begin{pmatrix} e^{-i(t+s)/2} & 0 \\ 0 & e^{i(t+s)/2} \end{pmatrix} \end{array}$$

- Physical implementation of $\mathbf{Z}_t$ by a continuous process:

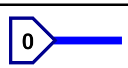
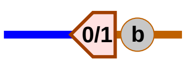
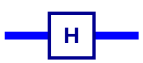
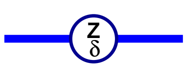
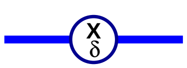


- $\mathbf{Z}_{t/N} = \mathbb{1} - i\frac{t}{N}\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} + O((t/N)^2).$

$$\mathbf{Z}_t = \lim_{N\to\infty}\left(\mathbb{1} - i\frac{t}{N}\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}\right)^N = e^{-i(\sigma_z/2)t}$$

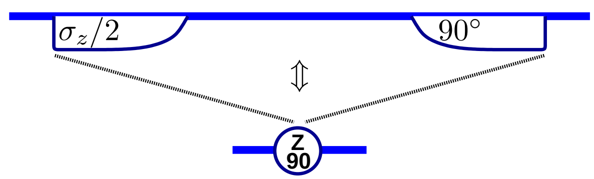
where $\sigma_z/2 \doteq \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ is the *generator* for $\mathbf{Z}$ rotations.

Summary of One-Qubit Gates

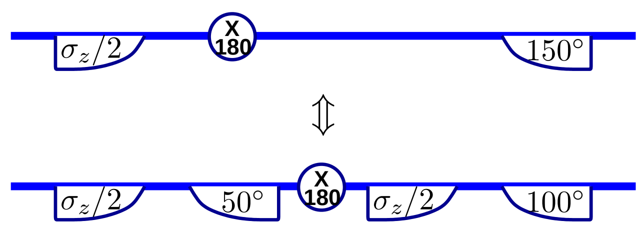
Gate picture	Symbol	Matrix form
	$\mathbf{prep}(\mathfrak{o})$	
	$\mathbf{meas}(Z\mapsto b)$	
	**not**	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
	**had**	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
	$\mathbf{Z}_\delta$	$\begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix}$
	$\mathbf{X}_\delta$	$\begin{pmatrix} \cos(\delta/2) & -i\sin(\delta/2) \\ -i\sin(\delta/2) & \cos(\delta/2) \end{pmatrix}$
	$\mathbf{Y}_\delta$	$\begin{pmatrix} \cos(\delta/2) & -\sin(\delta/2) \\ \sin(\delta/2) & \cos(\delta/2) \end{pmatrix}$

Continuously Evolving Qubits

- Network notation for continuous evolution:



- By default, gates are instantaneous.
  Relative scale matters:



Continuous Rotations Around Any Axis

- Rotation by δ around the $\hat{u}$ axis.

  $$\mathbf{rot}(\hat{u},\delta) = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\hat{u}\cdot\vec{\sigma}$$

  - One-parameter group: $\mathbf{rot}(\hat{u},\delta).\mathbf{rot}(\hat{u},\epsilon) = \mathbf{rot}(\hat{u},\delta+\epsilon)$.
  - Exponential form: $\mathbf{rot}(\hat{u},\delta) = e^{-i(\hat{u}\cdot\vec{\sigma}/2)\delta}$.
- Why is $e^{-i(\hat{u}\cdot\vec{\sigma}/2)\delta} = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\hat{u}\cdot\vec{\sigma}$?
  - Use $e^X = \mathbb{1} + X + X^2/2! + X^3/3! + X^4/4! + \ldots$ and $(-i\hat{u}\cdot\vec{\sigma})^k = (-i)^k(\hat{u}\cdot\vec{\sigma})^{k \bmod 2}$

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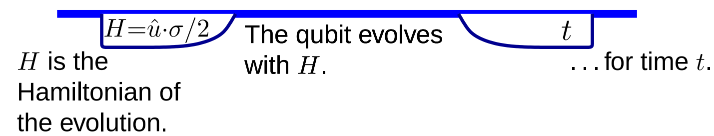
- Why is $e^{-i(\hat{u}\cdot\vec{\sigma}/2)\delta} = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\hat{u}\cdot\vec{\sigma}$?
  - Or use:
    1. $e^{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}} = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$
    2. So $e^{-i(\sigma_z/2)\delta} = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\sigma_z$.
    3. Choose $\hat{v}$ and ϵ so that $\mathbf{rot}(\hat{v},\epsilon)\sigma_z\mathbf{rot}(\hat{v},\epsilon) = \hat{u}\cdot\vec{\sigma}$.
    4. $UX^kU^\dagger = U\,XX\ldots U^\dagger = UXU^\dagger\,UXU^\dagger\,\ldots = (UXU^\dagger)^k$.
    5. $Ue^XU^\dagger = U(\mathbb{1} + X + X^2/2 + \ldots)U^\dagger$
       $= \mathbb{1} + UXU^\dagger + (UXU^\dagger)^2/2 + \ldots = e^{UXU^\dagger}$.
    6. $e^{-i(\hat{u}\cdot\vec{\sigma}/2)\delta} = \mathbf{rot}(\hat{v},\epsilon)e^{-i(\sigma_z/2)\delta}\mathbf{rot}(\hat{v},-\epsilon)$
       $= \mathbf{rot}(\hat{v},\epsilon)(\cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\sigma_z)\mathbf{rot}(\hat{v},-\epsilon)$
       $= \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\hat{u}\cdot\vec{\sigma}$

Continuous Rotations Around Any Axis

- Rotation by δ around the $\hat{u}$ axis.

  $$\mathbf{rot}(\hat{u},\delta) = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\hat{u}\cdot\vec{\sigma}$$

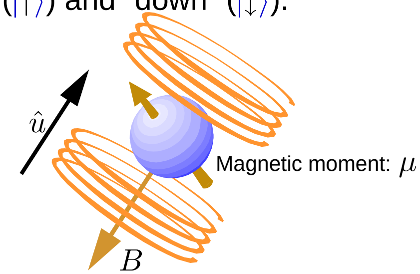
  - One-parameter group: $\mathbf{rot}(\hat{u},\delta).\mathbf{rot}(\hat{u},\epsilon) = \mathbf{rot}(\hat{u},\delta+\epsilon)$.
  - Exponential form: $\mathbf{rot}(\hat{u},\delta) = e^{-i(\hat{u}\cdot\vec{\sigma}/2)\delta}$.
- Why is $e^{-i(\hat{u}\cdot\vec{\sigma}/2)\delta} = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\hat{u}\cdot\vec{\sigma}$?
- $J_{\hat{u}} = \hat{u}\cdot\sigma/2$ is the spin operator along $\hat{u}$.



- The Hamiltonian is applied, or is part of the qubit's dynamics.
  - Note units: Energy units are angular frequency, $\hbar = 1$.

Example: Spin $1/2$ Qubit

- Spin $1/2$ in oriented space: One particle in a superposition of the states "up" ($|\uparrow\rangle$) and "down" ($|\downarrow\rangle$).

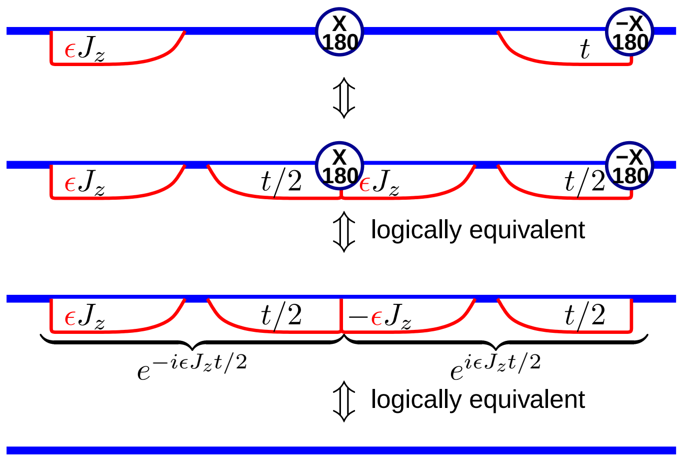


  - Apply a magnetic field in direction $-\hat{u}$ with strength B to cause the spin to evolve with Hamiltonian $B\mu J_{\hat{u}}$.

  ...in units where $\hbar = 1$.

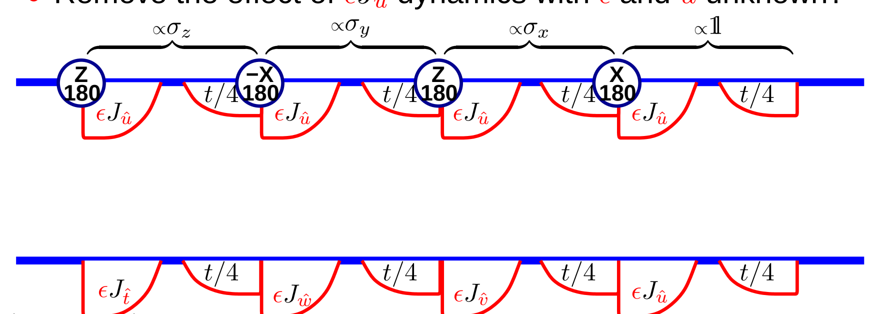
Application: Refocusing J_z

- How to remove the effect of ϵJ_z dynamics with ϵ unknown?



Application: Refocusing an Unknown Direction

- Remove the effect of $\epsilon J_{\hat{u}}$ dynamics with ϵ and $\hat{u}$ unknown?

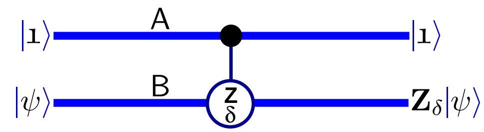


$$\Big(\mathbb{1} - i(t/4)\Big(- u_x J_x - u_y J_y + u_z J_z \\ - u_x J_x + u_y J_y - u_z J_z \\ u_x J_x - u_y J_y - u_z J_z \\ u_x J_x + u_y J_y + u_z J_z\Big) + O(t^2)\Big)$$

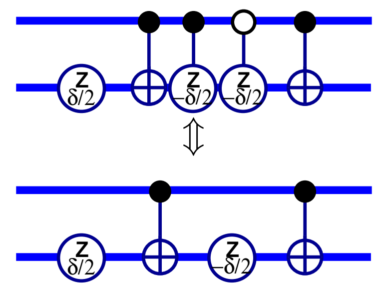
$\Updownarrow$ logically approximately equivalent up to $O(t^2)$

Conditional Z-Rotations

- Implementation of the conditional $\mathbf{Z}_\delta$ gate, $\mathbf{cZ}_\delta{}^{(AB)}$.

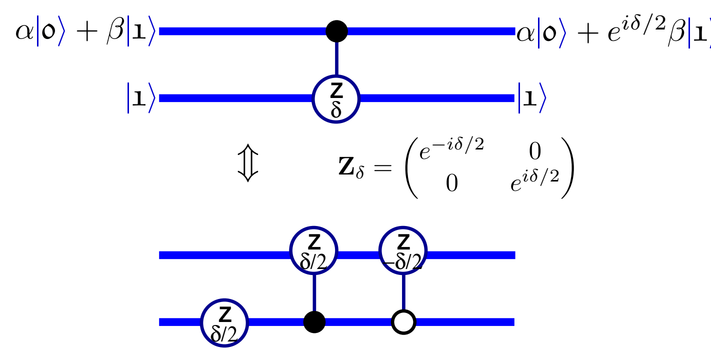


- Implementation with two cnots. $\Updownarrow$



Phase Kick-Back

- Conditional rotations conditionally kick back phases.



Phase Kick-Back

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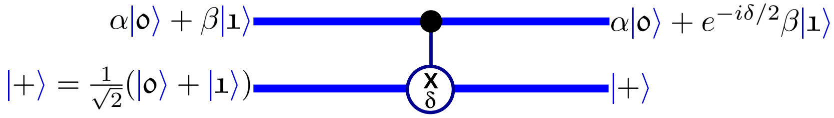
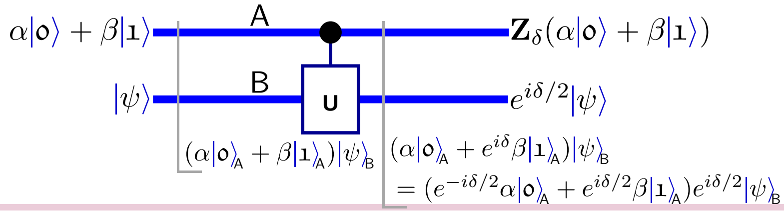


$$\mathbf{X}_\delta = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\sigma_x$$

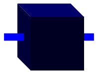
$$\mathbf{X}_\delta|+\rangle = (\cos(\delta/2) - i\sin(\delta/2))|+\rangle = e^{-i\delta/2}|+\rangle$$

- Phase kickback for any conditional operation.
  Suppose that $U|\psi\rangle = e^{i\delta}|\psi\rangle$.



The Rotation Angle Problem (RAP)

- Given: One-qubit device, a "black box".
  Promise: It applies $\mathbf{Z}_\theta$ for some unknown θ.
  Problem: Determine θ to within ϵ with high confidence.

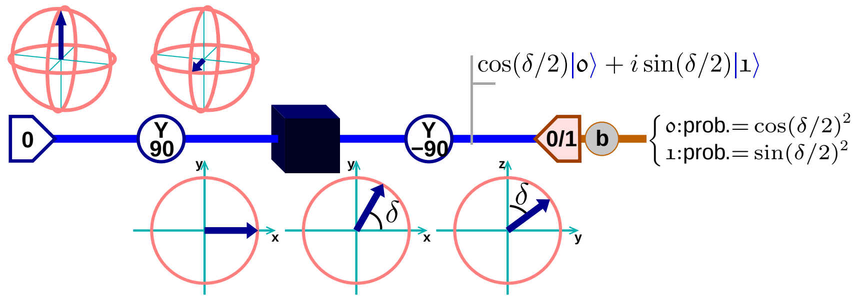


- Goal: Solve the problem using
  - $O(\frac{1}{\epsilon}\log\log(\frac{1}{\epsilon}))$ black-box applications ("queries").
  - $O(\log(\frac{1}{\epsilon})\log\log(\frac{1}{\epsilon}))$ one-qubit measurements.

- $O(f(\ldots))$ ("order of $f(\ldots)$") means
  "less than $Cf(\ldots)$ for some sufficiently large constant C".

RAP by Repeat Measurements?

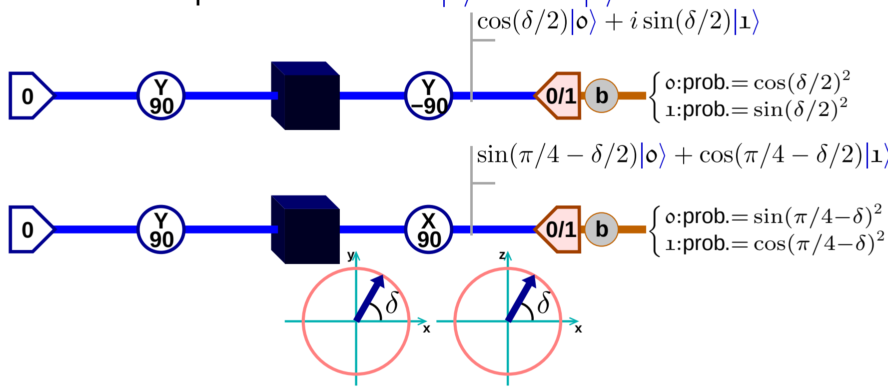
- Solve RAP by obtaining measurement statistics after modified queries that rotate $|0\rangle$ toward $|1\rangle$.



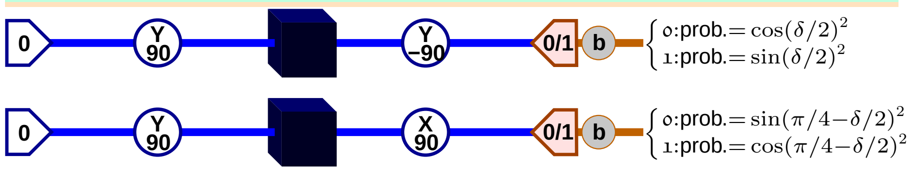
- Cannot distinguish between δ and $\delta + 180°$.

RAP by Repeat Measurements?

- Solve RAP by obtaining measurement statistics after modified queries that rotate $|0\rangle$ toward $|1\rangle$.



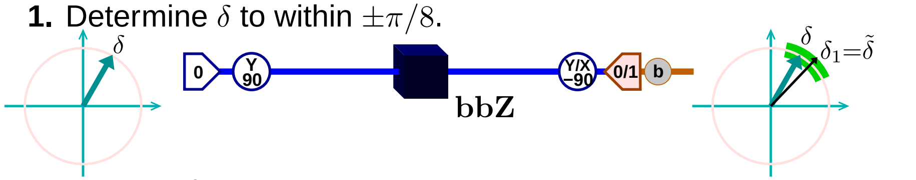
Measurement Statistics



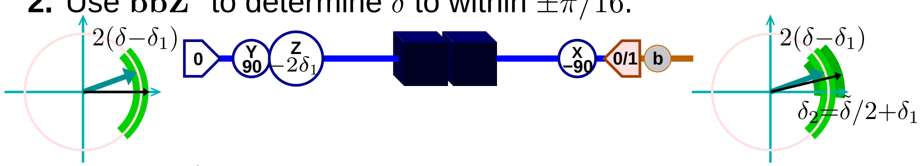
- Coin flip statistics for $\text{prob}(1) = p$, N trials:
  Expectation: $\langle p \rangle = p$.
  Variance: $v = p(1-p)/N$.
- The probability that p is more than Δ away from $\bar{p}$ is
  $$C(\Delta) \;<\; 2e^{-\Delta^2 N/2}$$
  Chernoff 1952 [1]
- N pairs of experiments yields $\tilde{\delta}$:
  $\tilde{\delta} \in \delta \pm \frac{\alpha}{\sqrt{N}}$ with probability $< 2e^{-\alpha^2/16}$.
- Need to improve accuracy and reduce measurement count.

RAP by Iteration

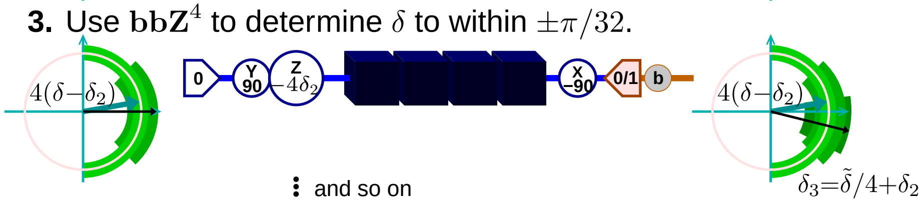
1. Determine δ to within $\pm\pi/8$.



2. Use $\mathbf{bbZ}^2$ to determine δ to within $\pm\pi/16$.

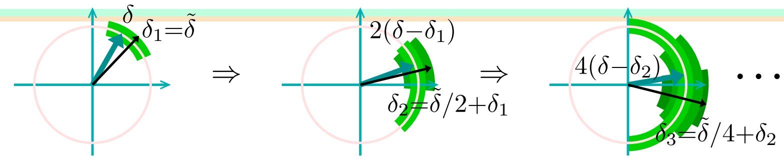


3. Use $\mathbf{bbZ}^4$ to determine δ to within $\pm\pi/32$.



⋮ and so on

RAP by Iteration: Resources



- Let N be the number of steps.
  Let k be the number of measurements in each step.
- Approximation: Obtain δ within $\epsilon = \pi/2^{N+2}$.
- Confidence: $> 1 - 2Ne^{-c_1 k}$ for some constant c_1.
  Confidence $1 - e^{-C}$ requires
  $$k > \log(2N/c_1) + C/c_1 = O(\log\log(\tfrac{1}{\epsilon})).$$
- Number of measurements: $kN = O(\log(\frac{1}{\epsilon})\log\log(\frac{1}{\epsilon}))$.
- Number of black box queries: $< k2^{N+1} = O(\frac{1}{\epsilon}\log\log(\frac{1}{\epsilon}))$

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References

[1] H. Chernoff. A measure of the asymptotic efficiency for tests of a hypothesis based on the sum of observations. *Ann. Math. Stat.*, 23:493–509, 1952.